Subharmonic and Superharmonic Resonances in the Pitch and Roll Modes of Ship Motions

Dean T. Mook,* Larry R. Marshall,† and Ali H. Nayfeh‡
Virginia Polytechnic Institute and State University, Blacksburg, Va.

We considered the motion of a ship with two degrees of freedom—pitch and roll—when second-order, static couplings are included in the equations of motion, and we obtained a nonlinear analysis of the response of the ship to a harmonic excitation (i.e., a regular sea) using the method of multiple scales. In the first part of the paper, we considered the pitch frequency to be twice the roll frequency and found that a superharmonic response exists for any value of the amplitude of the excitation, but a subharmonic response is possible only if the amplitude of the excitation is above a critical value. In the second part of the paper, we considered ships without the two-to-one frequency ratio. We found that there are five resonant situations and, depending on the value of the parameters, three can involve large motions. Some of the present results were combined with some results from a previous paper to determine the variation of the amplitude of the response with the encounter frequency. Also, the accuracy of some of the present results was established by comparing them with a numerical solution.

Introduction

IN an earlier paper[1] the authors considered the response of a ship which is free to roll and pitch only. Second-order static couplings between the two modes were included, and the frequency of small, free oscillations in pitch was taken to be twice the frequency of small, free oscillations in roll. The pitch and the roll response was determined for the two cases in which the excitation (encounter) frequency is either near the pitch frequency or near the roll frequency. There are other situations of special interest which result in either a superharmonic or a subharmonic response. In the first part of this paper, we determine the response in these two situations.

In the second part, we determine the response when there is no two-to-one frequency ratio. In this case, there are situations of special interest which contain superharmonic and subharmonic as well as combination responses.

The present results are then combined with those of the previous paper to give a complete picture of the various possible responses. When the complete picture is studied, the significance of the two-to-one frequency ratio can be seen clearly.

As in the first paper, we compare some of the results obtained by the asymptotic analysis with results obtained numerically.

Pitch and Roll Motions for Ships with the Pitch Frequency Twice the Roll Frequency.

Letting θ and ϕ denote the pitch and roll orientations, respectively, we write the equations of motion for a ship restrained to pitch and roll in a regular sea as follows:

$$I_{yy}\ddot{\theta} = M + M_w \cos(\Omega t + \tau_1) \tag{1a}$$

and

$$I_{xx}\ddot{\phi} = K + K_w \cos(\Omega t + \tau_2) \tag{1b}$$

where I_{xx} and I_{yy} are the moments of inertia of the ship, M and K are the pitch and roll moments due to the ship's oscillations in calm water, M_w and K_w are the amplitudes of the pitch and roll wave excitation moments, Ω is the wave encounter frequency, and τ_1 and τ_2 are phase angles. Also, we write

$$M = M_\theta\theta + M_q\theta + M_{\dot{q}}\ddot{\theta} + {}_{1/2}M_{\phi\phi}\phi^2 \tag{1c}$$

and

$$K = K_\phi\phi + K_p\dot{\phi} + K_{\dot{p}}\ddot{\phi} + K_{\phi\theta}\phi\,\theta \tag{1d}$$

where p and q are the roll and pitch rates. The coefficients appearing in Eqs. (1c) and (1d) as well as the pitch and roll wave excitation moments are frequency and speed dependent. Substituting for M and K in Eqs. (1a) and (1b) yields

$$\ddot{\theta} + \omega_1^2\theta = k_1\phi^2 - \hat{\mu}_1\dot{\theta} + F_1 \cos(\Omega t + \tau_1) \tag{1e}$$

and

$$\ddot{\phi} + \omega_2^2\phi = k_2\phi\theta - \hat{\mu}_2\dot{\phi} + F_2 \cos(\Omega t + \tau_2) \tag{1f}$$

where

$$(\omega_1^2, k_1, \hat{\mu}_1, F_1) = (I_{yy} - M_{\dot{q}})^{-1}[-M_\theta, {}_{1/2}M_{\phi\phi}, -M_q, M_w]$$

and

$$(\omega_2^2, k_2, \hat{\mu}_2, F_2) = (I_{xx} - K_{\dot{p}})^{-1}[-K_\phi, K_{\phi\theta}, -K_p, K_w].$$

It was shown in the first paper[1] that k_1 and k_2 must have the same sign so that constant-amplitude, periodic oscillations cannot exist when there is damping and no wave excitation. Thus, it is convenient to eliminate k_1 and k_2 from Eqs. (1e) and (1f) by letting

$$\bar{\theta} = k_2\theta \quad \text{and} \quad \bar{\phi} = (k_1k_2\phi)^{1/2} \tag{2}$$

Letting

$$\hat{f}_1 = k_2F_1 \text{ and } \hat{f}_2 = (k_1k_2)^{1/2}F_2 \tag{3}$$

and dropping the bars on θ and ϕ, we can rewrite Eqs. (1e) and (1f) as follows

$$\ddot{\theta} + \omega_1^2\theta = \phi^2 - \hat{\mu}_1\dot{\theta} + \hat{f}_1 \cos(\Omega t + \tau_1) \tag{4a}$$

Received June 4, 1973; revision received September 6, 1973.
Index category: Marine Hydrodynamics.
*Associate Professor. Member AIAA.
†Graduate Student and NDEA Fellow.
‡Professor. Member AIAA.

and

$$\ddot{\phi} + \omega_2^2\phi = \phi\theta - \hat{\mu}_2\dot{\phi} + f_2 \cos(\Omega t + \tau_2) \quad (4b)$$

To determine approximate solutions for θ and ϕ valid for small but finite amplitudes, we use the method of multiple scales.[2] Accordingly, different time scales are introduced by letting

$$T_n = \epsilon^n t;\ n = 0,1,\ldots \quad (5)$$

where ϵ is a measure of the response amplitudes. The time derivatives are transformed according to

$$d/dt = D_0 + \epsilon D_1 + \ldots \quad (6a)$$

and

$$d^2/dt^2 = D_0^2 + 2\epsilon D_0 D_1 + \ldots \quad (6b)$$

where $D_n = \partial/\partial T_n$. Moreover, θ and ϕ are assumed to have expansions of the form

$$\theta \sim \epsilon\theta_1(T_0,T_1) + \epsilon^2\theta_2(T_0,T_1) + \ldots \quad (7a)$$

and

$$\phi \sim \epsilon\phi_1(T_0,T_1) + \epsilon^2\phi_2(T_0,T_1) + \ldots \quad (7b)$$

The nearness of $2\omega_2$ to ω_1 is expressed by introducing the detuning parameter σ_1 so that

$$2\omega_2 = \omega_1 + \epsilon\sigma_1 \quad (8)$$

Also, the linear damping coefficients and the amplitudes of the excitation components are considered to be of the order of the response amplitudes; that is

$$\hat{\mu}_1 = \epsilon\mu_1,\ \hat{\mu}_2 = \epsilon\mu_2,\ \hat{f}_1 = \epsilon f_1,\ \hat{f}_2 = \epsilon f_2 \quad (9)$$

Substituting Eqs. (5-9) into Eqs. (4) and equating coefficients of equal powers of ϵ, we obtain

Order ϵ

$$D_0^2\theta_1 + \omega_1^2\theta_1 = \frac{1}{2}f_1 \exp[i(\Omega T_0 + \tau_1)] + cc \quad (10a)$$

$$D_0^2\phi_1 + \omega_2^2\phi_1 = \frac{1}{2}f_2 \exp[i(\Omega T_0 + \tau_2)] + cc \quad (10b)$$

Order ϵ^2

$$D_0^2\theta_2 + \omega_1^2\theta_2 = -2D_0D_1\theta_1 - \mu_1 D_0\theta_1 + \phi_1^2 \quad (11a)$$

$$D_0^2\phi_2 + \omega_2^2\phi_2 = -2D_0D_1\phi_1 - \mu_2 D_0\phi_1 + \theta_1\phi_1 \quad (11b)$$

where cc stands for the complex conjugate.

The solution of the first-order problem can be expressed as

$$\theta_1 = A_1(T_1)\exp(i\omega_1 T_0) + P_1\exp(i\Omega T_0) + cc \quad (12a)$$

$$\phi_1 = A_2(T_1)\exp(i\omega_2 T_0) + P_2\exp(i\Omega T_0) + cc \quad (12b)$$

where

$$P_m = \frac{1}{2}f_m \exp(i\tau_m)(\omega_m^2 - \Omega^2)^{-1} \quad (13)$$

Substituting for θ_1 and ϕ_1 from Eqs. (12), in Eqs. (11) yields

$$D_0^2\theta_2 + \omega_1^2\theta_2 = -i\omega_1(2A_1' + \mu_1 A_1)\exp(i\omega_1 T_0) - i\mu_1\Omega P_1\exp(i\Omega T_0) + A_2^2\exp[i(\omega_1 T_0 + \sigma_1 T_1)] + P_2^2\exp(i2\Omega T_0) + 2A_2P_2\exp[i(\Omega + \omega_2)T_0] + 2\bar{A}_2P_2\exp[i(\Omega - \omega_2)T_0] + A_2\bar{A}_2 + P_2\bar{P}_2 + cc \quad (14a)$$

and

$$D_0^2\phi_2 + \omega_2^2\phi_2 = -i\omega_2(2A_2' + \mu_2 A_2)\exp(i\omega_2 T_0) - i\mu_2\Omega P_2\exp(i\Omega T_0) + A_1A_2\exp[i(\omega_1 + \omega_2)T_0] + A_1\bar{A}_2\exp[i(\omega_2 T_0 - \sigma_1 T_1)] + A_2P_1\exp[i(\Omega + \omega_2)T_0] + \bar{A}_2P_1\exp[i(\Omega - \omega_2)T_0] + \bar{A}_1P_2\exp[i(\Omega - \omega_1)T_0] + A_1P_2\exp[i(\Omega + \omega_1)T_0] + P_1P_2\exp(i2\Omega T_0) + P_1\bar{P}_2 + cc \quad (14b)$$

where the prime indicates differentiation with respect to T_1.

The right-hand sides of Eqs. (14) contain terms proportional to $\exp(i\omega_n T_0)$. The corresponding particular solutions contain secular terms of the form $T_0 \exp(i\omega_n T_0)$ which make θ_2/θ_1 and ϕ_2/ϕ_1 unbounded as $T_0 \to \infty$ and thereby make the expansions given in Eqs. (7) invalid as $t \to \infty$. To obtain uniformly valid expansions, we need to eliminate all the terms on the right-hand sides which produce secular terms in the particular solutions.

Inspection of the right-hand sides of Eqs. (14) indicates that secular terms result from the nonlinear coupling terms when Ω is near $\omega_1/4$ (superharmonic resonance) and when Ω is near $3\omega_1/2$ (subharmonic resonance). These are the only two possibilities because it is required that Ω not be near ω_1 or ω_2. (These latter cases were considered in the first paper.)

A. Superharmonic Resonance

The nearness of Ω to $\omega_1/4$ is expressed by introducing a second detuning parameter σ_2 such that

$$\Omega = \frac{1}{4}\omega_1 + \epsilon\sigma_2 \quad (15)$$

Elimination of the terms on the right-hand sides of Eqs. (14) which produce secular terms in the expansions gives the following solvability conditions

$$-i\omega_1(2A_1' + \mu_1 A_1) + A_2^2\exp(i\sigma_1 T_1) = 0 \quad (16a)$$

and

$$-i\omega_2(2A_2' + \mu_2 A_2) + A_1\bar{A}_2\exp(-i\sigma_1 T_1) + \frac{1}{4}f_1f_2\exp\left\{i\left[\left(2\sigma_2 - \frac{1}{2}\sigma_1\right)T_1 + \tau_1 + \tau_2\right]\right\}(\omega_1^2 - \Omega^2)^{-1}(\omega_2^2 - \Omega^2)^{-1} = 0 \quad (16b)$$

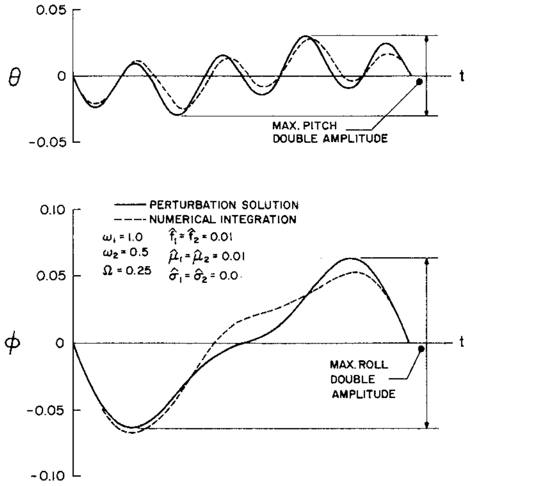

Fig. 1 Steady state superharmonic pitch and roll responses for one cycle of the wave encounter frequency.

We let

$$A_n = \frac{1}{2}a_n \exp(i\beta_n) \tag{17}$$

with a_n and β_n real, and define

$$\gamma_1 = 2\beta_2 - \beta_1 + \sigma_1 T_1 \tag{18}$$

and

$$\gamma_2 = \left(2\sigma_2 - \frac{1}{2}\sigma_1\right)T_1 - \beta_2 + \tau_1 + \tau_2 \tag{19}$$

Substituting Eqs. (17–19) into Eqs. (16) and then separating the real and imaginary parts, we have

$$a_1' = -\frac{1}{2}\mu_1 a_1 + \frac{1}{4}a_2^2\omega_1^{-1}\sin\gamma_1 \tag{20a}$$

$$a_1 a_2 \gamma_1' = \sigma_1 a_1 a_2 + \left(\frac{a_2^2}{4\omega_1} - \frac{a_1^2}{2\omega_2}\right)a_2\cos\gamma_1 - \frac{f_1 f_2 a_1 \cos\gamma_2}{2\omega_2(\omega_1^2-\Omega^2)(\omega_2^2-\Omega^2)} \tag{20b}$$

$$a_2' = -\frac{\mu_2}{2}a_2 - \frac{a_1 a_2}{4\omega_2}\sin\gamma_1 + \frac{f_1 f_2 \sin\gamma_2}{4\omega_2(\omega_1^2-\Omega^2)(\omega_2^2-\Omega^2)} \tag{20c}$$

and

$$a_2\gamma_2' = \left(2\sigma_2 - \frac{\sigma_1}{2}\right)a_2 + \frac{a_1 a_2}{4\omega_2}\cos\gamma_1 + \frac{f_1 f_2 \cos\gamma_2}{4\omega_2(\omega_1^2-\Omega^2)(\omega_2^2-\Omega^2)} \tag{20d}$$

For the steady-state response ($a_1' = a_2' = \gamma_1' = \gamma_2' = 0$), Eqs. (20) can be manipulated to yield

$$a_1^3 + 4\omega_2\left[\mu_2\sin\gamma_1 + 2\left(2\sigma_2 - \frac{1}{2}\sigma_1\right)\cos\gamma_1\right]a_1^2 + 4\omega_2^2\left[\mu_2^2 + 4\left(2\sigma_2 - \frac{1}{2}\sigma_1\right)^2\right]a_1 - \Lambda\{f_1 f_2(\omega_1^2-\Omega^2)^{-1}(\omega_2^2-\Omega^2)^{-1}\}^2 = 0 \tag{21a}$$

$$a_2 = (a_1/\Lambda)^{1/2},\ \sin\gamma_1 = 2\mu_1\omega_1\Lambda,\ \cos\gamma_1 = -16\sigma_2\omega_1\Lambda \tag{21b}$$

$$\sin\gamma_2 = (\omega_1^2-\Omega^2)(\omega_2^2-\Omega^2)(f_1 f_2)^{-1}(2\omega_2\mu_2 + a_1\sin\gamma_1)a_2, \tag{21c}$$

and

$$\cos\gamma_2 = -(\omega_1^2-\Omega^2)(\omega_2^2-\Omega^2)(f_1 f_2)^{-1}\left[4\omega_2\left(2\sigma_2 - \frac{1}{2}\sigma_1\right) + a_1\cos\gamma_1\right]a_2 \tag{21d}$$

where

$$\Lambda = [4\omega_1^2(\mu_1^2 + 64\sigma_2^2)]^{-1/2} \tag{21e}$$

Letting $\hat{a}_1 = \epsilon a_1$ and $\hat{a}_2 = \epsilon a_2$, we can write the first-order, steady-state, uniformly valid solutions for θ and ϕ as follows

$$\theta = \hat{f}_1(\omega_1^2-\Omega^2)^{-1}\cos(\Omega t + \tau_1) + \hat{a}_1\cos[4\Omega t - 2\gamma_2 - \gamma_1 + 2(\tau_1+\tau_2)] + 0(\epsilon^2) \tag{22a}$$

and

$$\phi = \hat{f}_2(\omega_2^2-\Omega^2)^{-1}\cos(\Omega t + \tau_2) + \hat{a}_2\cos[2\Omega t - \gamma_2 + \tau_1 + \tau_2] + 0(\epsilon^2) \tag{22b}$$

Equations (22) show that the effect of the superharmonic resonance is an entrainment of the otherwise-vanishing,

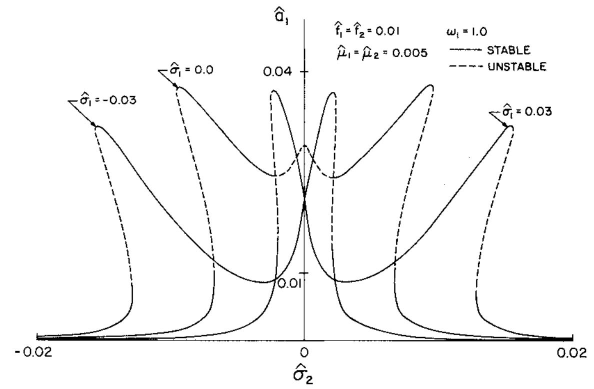

Fig. 2a The superharmonic amplitude $\hat{a}_1$ as a function of the detuning $\hat{\sigma}_2$ for various values of the detuning $\hat{\sigma}_1$.

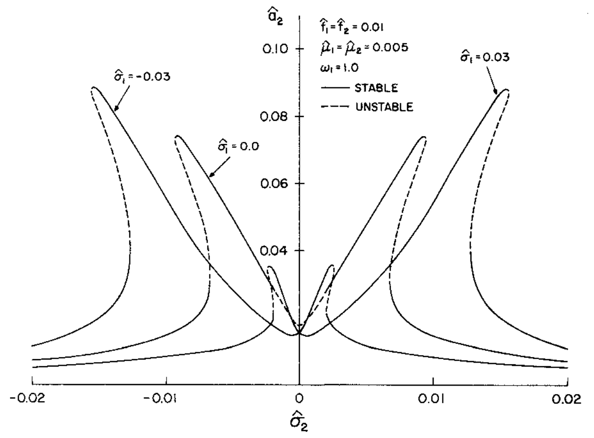

Fig. 2b The superharmonic amplitude $\hat{a}_2$ as a function of the detuning $\hat{\sigma}_2$ for various values of the detuning $\hat{\sigma}_1$.

homogeneous solution by the particular solution. The two combine to give pitch and roll responses that are sums of the two harmonics involved.

B. Subharmonic Resonance

The second detuning given by Eq. (15) is now redefined so that the nearness of Ω to $3\omega_1/2$ is expressed as

$$\Omega = \frac{3}{2}\omega_1 + \epsilon\sigma_2 \tag{23a}$$

Instead of Eq. (19), γ_2 is now defined as

$$\gamma_2 = \left(\sigma_2 - \frac{1}{2}\sigma_1\right)T_1 + \tau_2 - \beta_1 - \beta_2 \tag{23b}$$

Continuing to use the quantities defined in Eqs. (17) and (18) and following the same procedure detailed previously, we obtain

$$a_1' = -\frac{\mu_1}{2}a_1 + \frac{a_2^2}{4\omega_1}\sin\gamma_1 + \frac{f_2a_2}{2\omega_1(\omega_2^2-\Omega^2)}\sin\gamma_2 \tag{24a}$$

$$a_1a_2\gamma_1' = \sigma_1a_1a_2 + \left(\frac{a_2^2}{4\omega_1} - \frac{a_1^2}{2\omega_2}\right)a_2\cos\gamma_1 + \left(\frac{a_2^2}{2\omega_1} - \frac{a_1^2}{2\omega_2}\right)\frac{f_2\cos\gamma_2}{(\omega_2^2-\Omega^2)} \tag{24b}$$

$$a_2' = -\frac{\mu_2}{2}a_2 - \frac{a_1a_2}{4\omega_2}\sin\gamma_1 + \frac{f_2a_1\sin\gamma_2}{4\omega_2(\omega_2^2-\Omega^2)} \tag{24c}$$

and

$$a_1a_2\gamma_2' = \left(\sigma_2 - \frac{\sigma_1}{2}\right)a_1a_2 + \left(\frac{a_1^2}{4\omega_2} + \frac{a_2^2}{4\omega_1}\right)a_2\cos\gamma_1 + \left(\frac{a_1^2}{4\omega_2} + \frac{a_2^2}{2\omega_1}\right)\frac{f_2\cos\gamma_2}{(\omega_2^2-\Omega^2)} \tag{24d}$$

For the steady-state response, two possibilities exist

$$1)\ a_1 = a_2 = 0 \tag{25}$$

$$2)\ a_2^2 = \frac{5f_2^2}{4(\omega_2^2-\Omega^2)^2} - 4\omega_1\omega_2\left[\mu_1\mu_2 - \frac{8}{9}\sigma_2\left(\sigma_2 - \frac{3\sigma_1}{2}\right)\right] \pm \frac{f_2}{(\omega_2^2-\Omega^2)}\left\{\frac{9f_2^2}{4(\omega_2^2-\Omega^2)^2} - 4\omega_1\omega_2\left[\mu_1\mu_2 - 8\sigma_2\left(\sigma_2-\frac{3\sigma_1}{2}\right)\right] - \frac{64}{9}\omega_1^2\omega_2^2\left[\mu_1^2\left(\sigma_2 - \frac{3\sigma_1}{2}\right)^2 + 4\mu_2^2\sigma_2^2 + 4\mu_1\mu_2\sigma_2\left(\sigma_2 - \frac{3\sigma_1}{2}\right)\right]\right\}^{1/2} \tag{26a}$$

$$a_1^2 = \left\{3a_2^4 + \frac{12f_2^2a_2^2}{(\omega_2^2-\Omega^2)^2} - 16\omega_1\omega_2a_2^2\left[\mu_1\mu_2 - \frac{8}{3}\sigma_2\left(\sigma_2 - \frac{3\sigma_1}{2}\right)\right]\right\} \times [4(\mu_1^2 + 16\sigma_2^2)]^{-1} \tag{26b}$$

$$\sin\gamma_1 = \frac{4\omega_1\omega_2}{3a_1a_2^2}\left(\frac{\mu_1a_1^2}{2\omega_2} - \frac{\mu_2a_2^2}{\omega_1}\right) \tag{26c}$$

$$\cos\gamma_1 = \frac{8\omega_1\omega_2}{3a_1a_2^2}\left(\frac{\sigma_2a_1^2}{\omega_2} - \left(\sigma_2 - \frac{3\sigma_1}{2}\right)\frac{a_2^2}{\omega_1}\right) \tag{26d}$$

$$\sin\gamma_2 = \frac{\omega_1(\omega_2^2-\Omega^2)}{f_2a_2}\left(\mu_1a_1 - \frac{a_2^2}{2\omega_1}\sin\gamma_1\right) \tag{26e}$$

and

$$\cos\gamma_2 = -\frac{2\omega_1(\omega_2^2-\Omega^2)}{f_2a_2}\left(\frac{2}{3}\sigma_2a_1 + \frac{a_2^2}{4\omega_1}\cos\gamma_1\right). \tag{26f}$$

Letting $\hat{a}_1 = \epsilon a_1$ and $\hat{a}_2 = \epsilon a_2$, we can write the first-order uniformly valid solutions for θ and ϕ as follows

$$\theta = \hat{f}_1(\omega_1^2 - \Omega^2)^{-1}\cos(\Omega t + \tau_1) + \hat{a}_1\cos\left(\frac{2}{3}\Omega t - \frac{1}{3}\gamma_1 - \frac{2}{3}\gamma_2 + \frac{2}{3}\tau_2\right) + 0(\epsilon^2) \tag{27a}$$

and

$$\phi = \hat{f}_2(\omega_2^2 - \Omega^2)^{-1}\cos(\Omega t + \tau_2) + \hat{a}_2\cos\left(\frac{1}{3}\Omega t + \frac{1}{3}\gamma_1 - \frac{1}{3}\gamma_2 + \frac{1}{3}\tau_2\right) + 0(\epsilon^2) \tag{27b}$$

Eqs. (26a) and (26b) indicate that, depending on the values of the parameters involved, the roll excitation amplitude $\hat{f}_2$ will have to increase above a certain critical value before the homogeneous solution is entrained by the particular solution. This is found to be the case in the numerical example presented later.

C. Numerical Example

To illustrate the results for a specific ship or model, a consistent set of coefficients for Eqs. (1e) and (1f) is required. Because the authors did not possess this information, values for the coefficients were chosen arbitrarily and the responses calculated. Nevertheless, the results are useful in that the basic character of the solution is illustrated. Also, to verify the validity of the perturbation expansions developed herein, we integrated Eqs. (1e) and (1f) numerically using Hamming's Predictor-Corrector Method.

The pitch and roll responses for both the superharmonic- and subharmonic-resonant cases are sums of two harmonics [see Eqs. (22) and (27)]. A typical example of the steady-state superharmonic pitch and roll responses for one cycle of the wave encounter frequency is presented in Fig. 1.

In Figs. 2a and 2b, $\hat{a}_1$ and $\hat{a}_2$ for the superharmonic case are plotted as functions of $\hat{\sigma}_2$ for various values of $\hat{\sigma}_1$ with $\hat{f}_1 = \hat{f}_2 = 0.01$ and $\hat{\mu}_1 = \hat{\mu}_2 = 0.005$. An interesting feature is the so-called "jump" phenomenon. Figure 3 presents $\hat{a}_1$ as a function of $\hat{\sigma}_2$ for various values of the damping coefficients, $\hat{\mu}_1$ and $\hat{\mu}_2$, with $\hat{f}_1 = \hat{f}_2 = 0.01$ and $\hat{\sigma}_1 = 0.0$ for the superharmonic case. The extreme sensitivity of $\hat{a}_1$ (the situation is analogous for $\hat{a}_2$) to the amount of damping should be noted.

In the subharmonic case, Fig. 4 depicts $\hat{a}_1$ and $\hat{a}_2$ as functions of $\hat{f}_2$ for $\hat{\sigma}_2 = 0.05$. For the values of the parameters considered, $\hat{f}_2$ must increase beyond a critical value before the homogeneous solution adds to the response. Note that all solutions are unstable if $\hat{f}_2$ is greater than approximately 0.26. Figure 5 illustrates the severe sensitivity of $\hat{a}_2$ (the situation is analogous for $\hat{a}_1$) to the amount of damping.

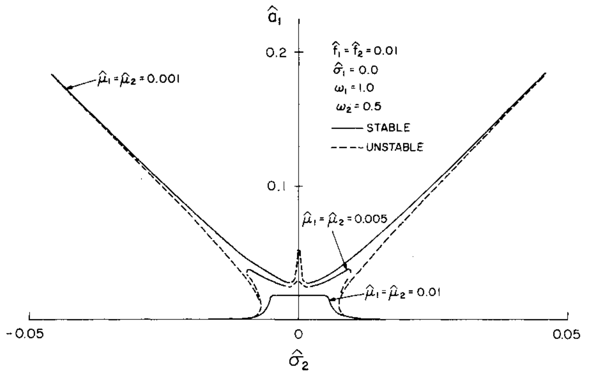

Fig. 3 The superharmonic amplitude $\hat{a}_1$ as a function of the detuning $\hat{\sigma}_2$ for various values of the damping coefficient.

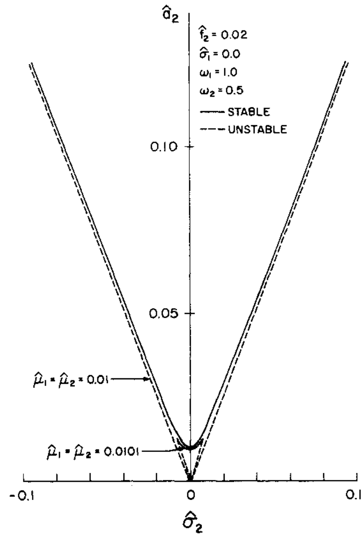

Fig. 5 The subharmonic amplitude $\hat{a}_2$ as a function of the detuning $\hat{\sigma}_2$ for two values of the damping coefficients.

Figures 6–8 present the maximum double amplitude of the pitch and the roll responses as functions of the wave encounter frequency for various values of $\hat{f}_1$ and $\hat{f}_2$ with $\hat{\mu}_1 = 0.1$, $\hat{\mu}_2 = 0.01$, $\omega_1 = 1.0$, and $\omega_2 = 0.5$. The responses when Ω is near ω_1 or ω_2 were computed according to the solutions given in the previous paper.[1]

In Fig. 6, the roll excitation amplitude $\hat{f}_2$ is larger than the pitch excitation amplitude $\hat{f}_1$; $\hat{f}_1 = 0.001$ and $\hat{f}_2 = 0.01$. This could possibly correspond to a beam-seas situation. The pitch and roll responses are largest when Ω is near ω_2. For this situation, the effect of the two-to-one frequency ratio ($\omega_1 \approx 2\omega_2$) is to transfer energy from the roll mode to the pitch mode.

In Fig. 7, $\hat{f}_1 = \hat{f}_2 = 0.01$ and this might be illustrative of a quartering seas situation. Large responses for both pitch and roll are predicted when Ω is near ω_1 and ω_2. Notice that the superharmonic resonance (i.e., when $\Omega = 0.25$) produces responses that are approximately fifty % greater than the linear responses. The effect of the two-to-one frequency ratio here is to transfer energy from the roll mode to the pitch mode when $\Omega \approx \omega_2$ and to transfer energy from the pitch mode to the roll mode when $\Omega \approx \omega_1$.

In Fig. 8, $\hat{f}_1$ is larger than $\hat{f}_2$ ($\hat{f}_1 = 0.02$ and $\hat{f}_2 = 0.001$) and this could model a head or following seas situation. The pitch response is largest when $\Omega \approx \omega_1$, which is expected. However, the roll response is also the largest when $\Omega \approx \omega_1$, which is contrary to what is expected from linear theory. This is a consequence of the "saturation" phenomenon noted in Ref. 1.

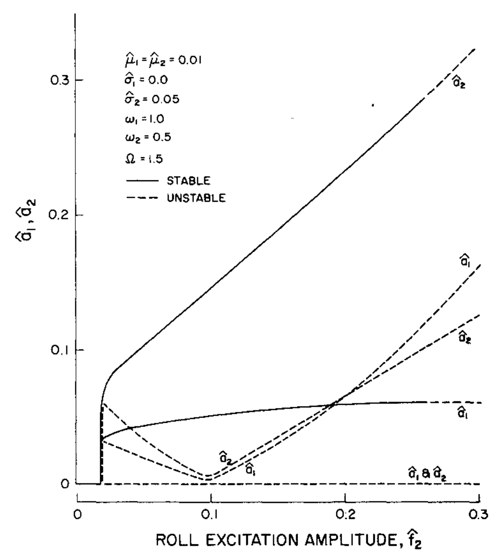

Fig. 4 The subharmonic amplitudes, $\hat{a}_1$ and $\hat{a}_2$, as functions of the roll excitation amplitude for $\hat{\sigma}_2 = 0.05$.

For the forcing amplitudes considered in Figs. 6–8, the superharmonic resonance adds only slightly to the linear response while the subharmonic resonance is not manifested at all. This is because the roll excitation amplitudes chosen are below the critical value required to produce subharmonic responses. Had the value of $\hat{f}_2$ been above the critical value, the response near the roll natural frequency (ω_2) would have become extremely large but the subharmonic response still relatively small. It is noted, however, that the subharmonic resonance can produce large amplitude motions when $\hat{f}_2$ is greater than its critical value. In fact, for $\hat{f}_2$ sufficiently large, the motions become unstable as shown in Fig. 4.

The significance of the two-to-one frequency ratio can be illustrated by comparing these responses with those for a ship without the frequency ratio. The responses for such a ship are obtained in the next section and then, with the aid of a numerical example, a comparison is made.

The stability of the steady-state solutions were determined as in Ref. 1

Pitch and Roll Motions for Ships with the Pitch Frequency not near Twice the Roll Frequency

With ω_1 not near $2\omega_2$ and Ω near either ω_1 or ω_2, there is no coupling between the pitch and roll modes to first-order and the responses are given by the linear solutions. To ascertain the resonant situations that can occur when ω_1 is not near $2\omega_2$ and Ω is not near either ω_1 or ω_2, we

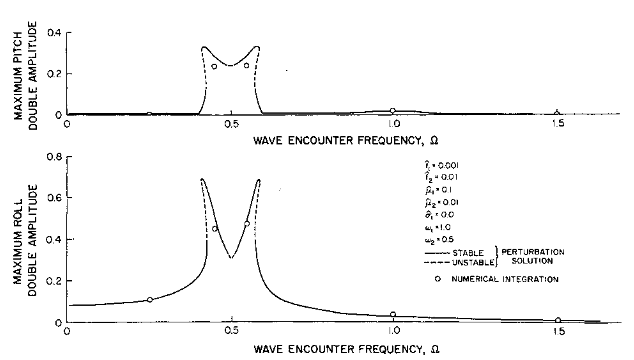

Fig. 6 Maximum pitch and roll double amplitudes as functions of the wave encounter frequency with $\hat{f}_1 = 0.001$ and $\hat{f}_2 = 0.01$ for ships with the two-to-one frequency ratio.

rewrite Eqs. (14) as

$$D_0^2\theta_2 + \omega_1^2\theta_2 = -i\omega_1(2A_1' + \mu_1A_1)\exp(i\omega T_0) - i\mu_1\Omega P_1\exp(i\Omega T_0) + A_2^2\exp(i2\omega_2T_0) + P_2^2\exp(i2\Omega T_0) + 2A_2P_2\exp[i(\Omega + \omega_2)T_0] + 2\bar{A}_2P_2\exp[i(\Omega - \omega_2)T_0] + A_2\bar{A}_2 + P_2\bar{P}_2 + cc \quad (28a)$$

and

$$D_0^2\phi_2 + \omega_2^2\phi_2 = -i\omega_2(2A_2' + \mu_2A_2)\exp(i\omega_2T_0) - i\mu_2\Omega P_2\exp(i\Omega T_0) + A_1A_2\exp[i(\omega_1 + \omega_2)T_0] + A_1\bar{A}_2\exp[i(\omega_1 - \omega_2)T_0] + A_2P_1\exp[i(\Omega + \omega_2)T_0] + \bar{A}_2P_1\exp[i(\Omega - \omega_2)T_0] + \bar{A}_1P_2\exp[i(\Omega - \omega_1)T_0] + A_1P_2\exp[i(\Omega + \omega_1)T_0] + P_1P_2\exp(i2\Omega T_0) + P_1\bar{P}_2 + cc \quad (28b)$$

There are five possibilities when we restrict ω_1 to be greater than ω_2: $\Omega \approx \omega_1/2$, $\Omega \approx \omega_2/2$, $\Omega \approx 2\omega_2$, $\Omega \approx \omega_1 + \omega_2$, and $\Omega \approx \omega_1 - \omega_2$. These cases are treated separately below.

A. The Case of Ω Near $\omega_1/2$

Let the nearness of Ω to $\omega_1/2$ be expressed as

$$\Omega = (1/2)\omega_1 + \epsilon\sigma \quad (29)$$

where σ is the detuning. The solvability conditions are

$$-i\omega_1(2A_1' + \mu_1A_1) + \frac{1}{4}f_2^2(\omega_2^2 - \Omega^2)^{-2}\exp[i(2\sigma T_1 + 2\tau_2)] = 0 \quad (30a)$$

and

$$-i\omega_2(2A_2' + \mu_2A_2) = 0 \quad (30b)$$

Eqs. (30) can be solved separately since they are uncoupled. The solution of Eq. (30b) can be written as

$$A_2 = a_0\exp\left(-\frac{1}{2}\mu_2T_1\right)\exp(i\beta_0) + cc \quad (31)$$

where a_0 and β_0 are constants determined from initial conditions. As $T_1 \to \infty$, A_2 is damped out and the steady-state solution for the roll mode is

$$\phi = \hat{f}_2(\omega_2^2 - \Omega^2)^{-1}\cos(\Omega t + \tau_2) + O(\epsilon^2). \quad (32)$$

To analyze Eq. (30a), we let

$$A_1 = a\exp[i2(\sigma T_1 + \tau_2)] \quad (33)$$

where a is complex and obtain

$$-i2\omega_1a' + 4\omega_1\sigma a - i\omega_1\mu_1a + P_2^2 = 0 \quad (34)$$

Putting $a = a_r + ia_i$ with real a_r and a_i into Eq. (34) and separating the real and imaginary parts, we obtain

$$2a_r' + \mu_1a_r - 4\sigma a_i = 0 \quad (35a)$$

and

$$2a_i' + \mu_1a_i + 4\sigma a_r + f_2^2(4\omega_1)^{-1}(\omega_2^2 - \Omega^2)^{-2} = 0 \quad (35b)$$

The homogeneous solution of Eqs. (35) is given by

$$(a_r)_h = b_1\exp(mT_1) + cc \text{ and } (a_i)_h = b_2\exp(mT_1) + cc \quad (36a)$$

where b_1 and b_2 are related arbitrary constants and

$$m = -\frac{1}{2}\mu_1 \pm i2\sigma \quad (36b)$$

The particular solution of Eqs. (35) is

$$(a_r)_p = -\sigma f_2^2[\omega_1(\mu_1^2 + 16\sigma^2)]^{-1}(\omega_2^2 - \Omega^2)^{-2} \quad (37a)$$

and

$$(a_i)_p = -\mu_1f_2^2[4\omega_1(\mu_1^2 + 16\sigma^2)]^{-1}(\omega_2^2 - \Omega^2)^{-2} \quad (37b)$$

Since the real part of m is negative, the homogeneous solution of Eqs. (35) is damped out as $T_1 \to \infty$ and the steady-state response is given by the particular solution. The steady-state solution for the pitch mode can now be written as

$$\theta = \hat{f}_1(\omega_1^2 - \Omega^2)^{-1}\cos(\Omega t + \tau_1) + \hat{f}_2^2(2\omega_1)^{-1}(\omega_2^2 - \Omega^2)^{-2}(\hat{\mu}_1^2 + 16\hat{\sigma}^2)^{-1/2}\cos(2\Omega t + 2\tau_2 + \delta) + O(\epsilon^2) \quad (38a)$$

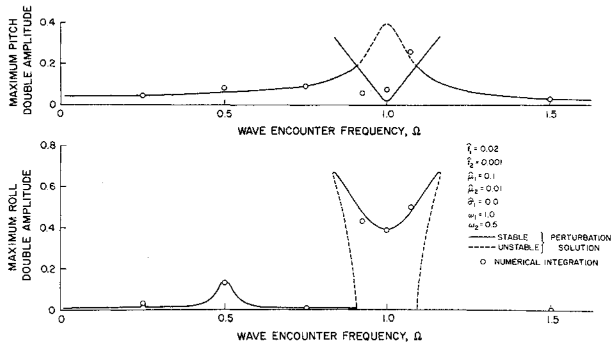

Fig. 8 Maximum pitch and roll double amplitudes as functions of the wave encounter frequency with $\hat{f}$ = 0.02 and $\hat{f}_2$ = 0.001 for ships with the two-to-one frequency ratio.

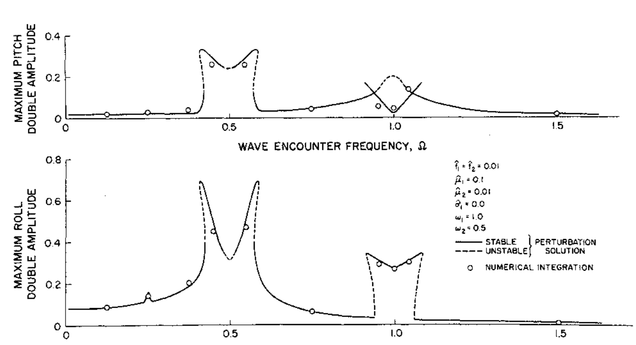

Fig. 7 Maximum ptich and roll double amplitudes as functions of the wave encounter frequency with $\hat{f}_1 = \hat{f}_2$ = 0.01 for ships with the two-to-one frequency ratio.

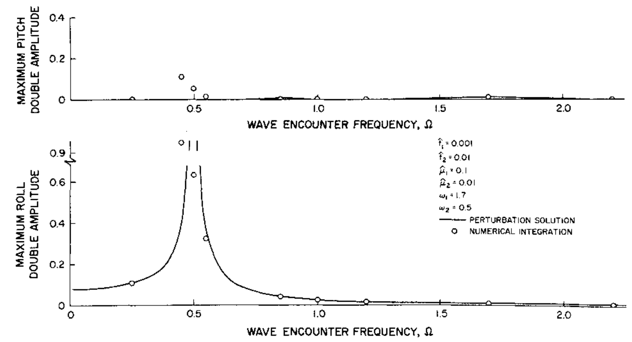

Fig. 9 Maximum pitch and roll double amplitudes as functions of the wave encounter frequency with $\hat{f}_1 = 0.001$ and $\hat{f}_2 = 0.01$ for ships not possessing the two-to-one frequency ratio.

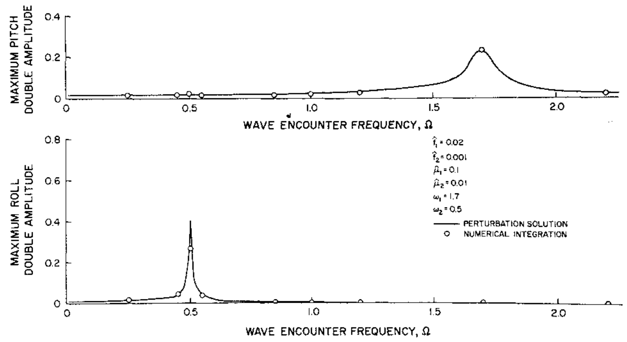

Fig. 11 Maximum pitch and roll double amplitudes as functions of the wave encounter frequency with $\hat{f}_1 = 0.02$ and $\hat{f}_2 = 0.001$ for ships not possessing the two-to-one frequency ratio.

where

$$\delta = \tan^{-1}\left(\frac{\hat{\mu}_1}{4\hat{\sigma}}\right) \quad (38b)$$

B. The Case of Ω Near $\omega_2/2$

The nearness of Ω to $\omega_2/2$ is expressed as

$$\Omega = \frac{1}{2}\omega_2 + \epsilon\sigma \quad (39)$$

Following the same procedure outlined in the previous case, we find that the steady-state pitch and roll responses are:

$$\theta = \hat{f}_1(\omega_1{}^2 - \Omega^2)^{-1}\cos(\Omega t + \tau_1) + 0(\epsilon^2) \quad (40a)$$

and

$$\phi = \hat{f}_2(\omega_2{}^2 - \Omega^2)^{-1}\cos(\Omega t + \tau_2) + \hat{f}_1\hat{f}_2(2\omega_2)^{-1}(\omega_1{}^2 - \Omega^2)^{-1}(\omega_2{}^2 - \Omega^2)^{-1}(\hat{\mu}_2{}^2 + 16\sigma^2)^{-1/2}\cos(2\Omega t + \tau_1 + \tau_2 + \delta) + 0(\epsilon^2) \quad (40b)$$

where

$$\delta = \tan^{-1}\left(\frac{\hat{\mu}_2}{4\hat{\sigma}}\right) \quad (40c)$$

C. The Case of Ω Near $2\omega_2$

The nearness of Ω to $2\omega_2$ is expressed as

$$\Omega = 2\omega_2 + \epsilon\sigma \quad (41)$$

and hence the solvability condition from the roll equation,

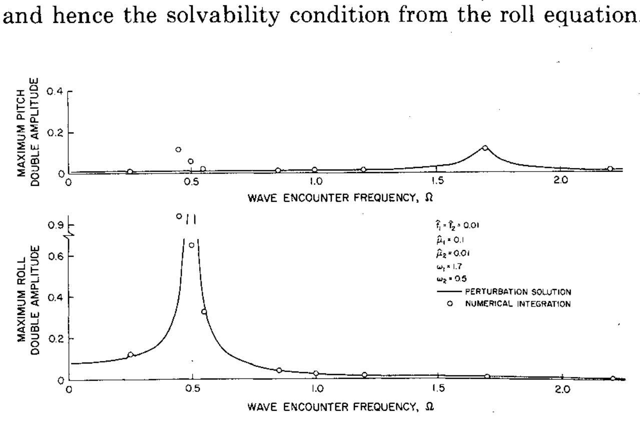

Fig. 10 Maximum pitch and roll double amplitudes as functions of the wave encounter frequency with $\hat{f}_1 = \hat{f}_2 = 0.01$ for ships not possessing the two-to-one frequency ratio.

Eq. (28b), becomes

$$-i\omega_2(2A_2{}' + \mu_2 A_2) + \frac{1}{2}\bar{A}_2 f_1(\omega_1{}^2 - \Omega^2)^{-1}\exp[i(\sigma T_1 + \tau_1)] = 0 \quad (42)$$

We let

$$A_2 = a\exp\left[i\left(\frac{\sigma}{2}T_1 + \frac{\tau_1}{2}\right)\right] \quad (43)$$

where a is complex and obtain

$$-2ia' + \sigma a - i\mu_2 a + \frac{1}{2}f_1\bar{a}[\omega_2(\omega_1{}^2 - \Omega^2)]^{-1} = 0 \quad (44)$$

Putting $a = a_r + ia_i$ with real a_r and a_i into Eq. (46) and separating the real and imaginary parts, we obtain

$$2a_r{}' + \mu_2 a_r - \sigma a_i + \frac{1}{2}f_1 a_i[\omega_2(\omega_1{}^2 - \Omega^2)]^{-1} = 0 \quad (45a)$$

and

$$2a_i{}' + \mu_2 a_i + \sigma a_r + \frac{1}{2}f_1 a_r[\omega_2(\omega_1{}^2 - \Omega^2)]^{-1} = 0 \quad (45b)$$

The solution of Eqs. (45) can be written as

$$a_r = b_1\exp(mT_1) + cc \quad \text{and} \quad a_i = b_2\exp(mT_1) + cc \quad (46a)$$

where b_1 and b_2 are related arbitrary constants and

$$m = -\frac{\mu_2}{2} \pm \frac{1}{2}\left[\frac{f_1{}^2}{4\omega_2{}^2(\omega_1{}^2 - \Omega^2)^2} - \sigma^2\right]^{1/2} \quad (46b)$$

Equation (46b) indicates that the real part of one value of m is always negative and hence the steady-state rolling motions resulting from the first term of Eq. (13b) are damped out. The second value of m is positive if

$$f_1 > 2\omega_2|\omega_1{}^2 - \Omega^2|(\sigma^2 + \mu_2{}^2)^{1/2} \quad (47)$$

If m is positive, unstable motions result in the sense that the amplitude of roll is no longer small and the present analysis becomes invalid while attempting to predict these amplitudes.

For stable motions, i.e., the real parts of both values of m are negative, the steady-state pitch and roll responses can be written as

$$\theta = \hat{f}_1(\omega_1{}^2 - \Omega^2)^{-1}\cos(\Omega t + \tau_1) + 0(\epsilon^2) \quad (48a)$$

and

$$\phi = \hat{f}_2(\omega_2{}^2 - \Omega^2)^{-1}\cos(\Omega t + \tau_2) + 0(\epsilon^2) \quad (48b)$$

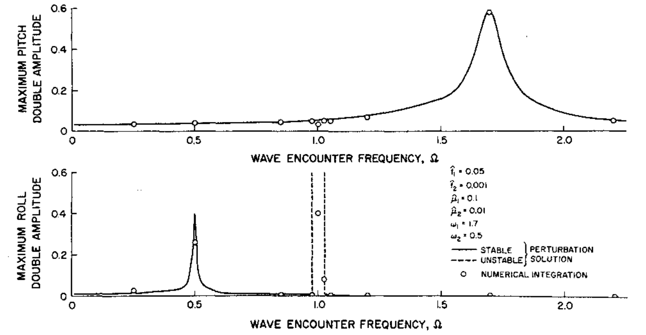

Fig. 12 Maximum pitch and roll double amplitudes as functions of the wave encounter frequency with $\hat{f}_1 = 0.05$ and $\hat{f}_2 = 0.001$ for ships not possessing the two-to-one frequency ratio.

D. The Case of Ω near $(\omega_1 + \omega_2)$

The nearness of Ω to $(\omega_1 + \omega_2)$ is expressed as

$$\Omega = \omega_1 + \omega_2 + \epsilon\sigma \tag{49}$$

The solvability conditions are

$$-i\omega_1(2A_1' + \mu_1 A_1) + \bar{A}_2 f_2 \exp[i(\sigma T_1 + \tau_2)](\omega_2^2 - \Omega^2)^{-1} = 0 \tag{50a}$$

and

$$-i\omega_2(2A_2' + \mu_2 A_2) + \frac{1}{2}\bar{A}_1 f_2(\omega_2^2 - \Omega^2)^{-1}\exp[i(\sigma T_1 + \tau_2)] = 0 \tag{50b}$$

We let

$$A_1 = a_1 \exp\left[i\left(\frac{1}{2}\sigma T_1 + \tau_2\right)\right] \text{and } A_2 = a_2 \exp\left(\frac{1}{2}i\sigma T_1\right) \tag{51}$$

where a_1 and a_2 are complex and obtain

$$-2ia_1' + \sigma a_1 - i\mu_1 a_1 + f_2\bar{a}_2[\omega_1(\omega_2^2 - \Omega^2)]^{-1} = 0 \tag{52a}$$

and

$$-2ia_2' + \sigma a_2 - i\mu_2 a_2 + f_2\bar{a}_1[2\omega_2(\omega_2^2 - \Omega^2)]^{-1} = 0 \tag{52b}$$

Putting $a_1 = a_{1r} + ia_{1i}$ and $a_2 = a_{2r} + ia_{2i}$ with real a_{1r}, a_{1i}, a_{2r}, and a_{2i}, and separating the real and imaginary parts, we have

$$2a_{1r}' + \mu_1 a_{1r} - \sigma a_{1i} + f_2 a_{2i}[\omega_1(\omega_2^2 - \Omega^2)]^{-1} = 0 \tag{53a}$$

$$2a_{1i}' + \mu_1 a_{1i} + \sigma a_{1r} + f_2 a_{2r}[\omega_1(\omega_2^2 - \Omega^2)]^{-1} = 0 \tag{53b}$$

$$2a_{2r}' + \mu_2 a_{2r} - \sigma a_{2i} + f_2 a_{1i}[2\omega_2(\omega_2^2 - \Omega^2)]^{-1} = 0 \tag{53c}$$

and

$$2a_{2i}' + \mu_2 a_{2i} + \sigma a_{2r} + f_2 a_{1r}[2\omega_2(\omega_2^2 - \Omega^2)]^{-1} = 0 \tag{53d}$$

The solution of Eqs. (53) is of the form

$$a_{1r} = b_1 \exp(mT_1) + cc \quad \text{and} \quad a_{1i} = b_2 \exp(mT_1) + cc \tag{54a}$$

$$a_{2r} = b_3 \exp(mT_1) + cc \quad \text{and} \quad a_{2i} = b_4 \exp(mT_1) + cc \tag{54b}$$

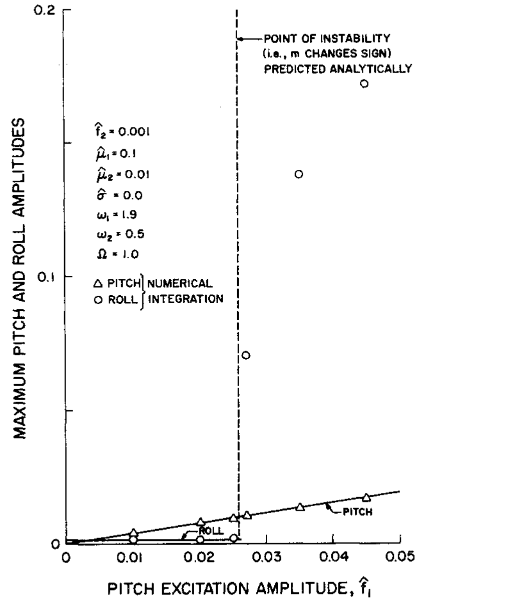

Fig. 13 Maximum pitch and roll amplitudes as functions of the pitch excitation amplitude.

where b_1, b_2, b_3, and b_4 are related arbitrary constants and m is given by

$$m = -\left[\frac{\mu_1 + \mu_2}{2}\right] \pm \left\{\left[\frac{\mu_1 + \mu_2}{2}\right]^2 - \frac{1}{4}\left[\mu_1\mu_2 + \sigma^2 - \frac{f_2^2}{2\omega_1\omega_2(\omega_2^2 - \Omega^2)^2}\right] \pm \frac{i\sigma}{4}(\mu_1 - \mu_2)\right\}^{1/2} \tag{55}$$

All the real parts of the four values of m determined from Eq. (55) must be negative for the motions to be stable and hence for the first terms of Eqs. (12) to be damped out. If not, the motions are unstable and can become large. For stable motions, the steady-state pitch and roll responses are given by Eqs. (48).

E. The Case of Ω near $(\omega_1 - \omega_2)$

The nearness of Ω to $(\omega_1 - \omega_2)$ is expressed as

$$\Omega = \omega_1 - \omega_2 + \epsilon\sigma \tag{56}$$

Following the same procedure as in Case D leads to the following relation for determining the stability of the motion

$$m = -\left[\frac{\mu_1 + \mu_2}{4}\right] \pm \left\{\left[\frac{\mu_1 + \mu_2}{4}\right]^2 - \frac{1}{4}\left[\mu_1\mu_2 + \sigma^2 + \frac{f_2^2}{2\omega_1\omega_2(\omega_2^2 - \Omega^2)^2}\right] \pm \frac{i\sigma}{4}(\mu_1 - \mu_2)\right\}^{1/2} \tag{57}$$

When $\sigma = 0$, the real part of m is always negative and hence the motions are stable. As σ or f_2 become large, the real part of m is also negative. For stable motions, the steady-state pitch and roll response is given by Eqs. (48).

F. Numerical Example

We chose $\omega_1 = 1.7$, $\omega_2 = 0.5$, and the values of $\hat{\mu}_1$ and $\hat{\mu}_2$ to agree with those in the previous numerical example. The values of $\hat{f}$ and $\hat{f}$ in Figs. 9–11 agree with those in

Figs. 6–8, respectively. Hence, comparisons of the corresponding pairs of figures illustrate the effects of the internal resonance (the two-to-one frequency ratio). A comparison of Figs. 6 and 9 reveals that the internal resonance can lead to a transfer of energy from roll to pitch and hence to a reduction in the roll amplitude. A comparison of Figs. 7 and 10 as well as Figs. 8 and 11 reveals that the internal resonance broadens the band of frequencies for which large roll amplitudes develop and that without the internal resonance there is no saturation phenomenon. That is, without internal resonance, there are no situations in which large amounts of energy can be fed into the roll made when the excitation frequency is near the pitch frequency. The values of $\hat{f}_1$ and $\hat{f}_2$ are below the critical value and hence there are no unstable situations of the type discussed in parts C, D and E above.

In Fig. 12, $\hat{f}_1$ is above the critical value indicated in Eq. (47) so that for a band of frequency around $\Omega = 1$ there is unstable motion. In order to see the character of the instability better, we have plotted numerical results in Fig. 13 and indicated the point at which the perturbation analysis predicts instability. Figure 13 corresponds to part C. The motion is actually bounded, as the numerical integration indicates.

When the values of $\hat{f}_2$ are large enough to cause unstable motion near $\Omega = 2.2$, the response near $\Omega = 0.5$ is so large that the asymptotic results no longer accurately predict the response.

Conclusions

1) With internal resonance, a superharmonic response will develop for all amplitudes of the excitation, but a subharmonic response will develop only if the amplitude of the excitation is above a certain value and below another. Jump phenomena exist for both cases. When the whole frequency spectrum is considered, one sees that there is a saturation phenomenon associated with the response in which large amounts of energy can be fed into the roll mode when the excitation frequency is near the pitch frequency. (This is discussed in detail in Ref. 1.)

2) Without internal resonance, there is neither jump nor saturation phenomena associated with the response. There are bands of frequency away from the roll and pitch frequencies in which large roll, and sometimes pitch, amplitudes will develop when the excitation amplitude is above a critical value.

3) There are more frequencies at which large roll amplitudes can exist for the case with internal resonance than for the case without it. These frequencies are uniquely determined by the coefficients of the undamped, unforced linear oscillation problem. Hence, specific numerical values for the linear damping, forcing and nonlinear terms in the equations of motion are not needed to ascertain the resonant situations that can lead to undesirable motions of ships. This is an outstanding feature of this analysis. This suggests that, when the full six-degree-of-freedom problem is considered, a designer can determine all the undesirable encounter frequencies for a given ship with no more information than what is required to solve the linear problem.

4) Generally, the accuracy of the asymptotic expansions in this paper and in Ref. 1 is well established by the numerical integration. This illustrates that the method of multiple scales is a powerful tool that can be used to analyze other problems of this nature.

5) Combination responses (sums of two harmonics) are possible in both pitch and roll for ships with or without the internal resonance. This can be observed from records of the pitching and rolling motions of actual ships at sea.[3] The records presented in this reference also show a transfer-of-energy phenomenon similar to that which exists between pitch and roll near perfect tuning for the superharmonic resonance presented in this paper.

6) The results of this paper and those presented in Ref. 1 indicate that the steady-state pitch and roll response frequency of ships is not always the same as the wave encounter frequency. Since the coefficients in the governing equations are frequency dependent, the coefficients in each equation must be computed to correspond to the oscillation frequency of that mode.

7) The response for other commensurable values of ω_1, ω_2, and Ω can be obtained using the same general procedure.

References

[1]Nayfeh, A. H., Mook, D. T., and Marshall, L. R., "Nonlinear Coupling of Pitch and Roll Modes in Ship Motions," V.P.I. Report 12, May 1973, Virginia Polytechnic Institute and State Univ., Blacksburg, Va., also *Journal of Hydronautics,* Vol 7, No. 4, Oct. 1973, pp. 145–152.

[2]Nayfeh, A. H., *Perturbation Methods,* Wiley-Interscience, New York, 1973, Chap. 6.

[3]Robb, A. M., *Theory of Naval Architecture,* Charles Griffin and Co. Ltd., London, England, 1952, Chap. 18.